

Adapted numerical methods for the numerical solution of the Poisson equation with L^2 boundary data in non-convex domains*

Thomas Apel[†] Serge Nicaise[‡] Johannes Pfefferer[§]

February 18, 2016

Abstract The very weak solution of the Poisson equation with L^2 boundary data is defined by the method of transposition. The finite element solution with regularized boundary data converges in the $L^2(\Omega)$ -norm with order $1/2$ in convex domains but has a reduced convergence order in non-convex domains although the solution remains to be contained in $H^{1/2}(\Omega)$. The reason is a singularity in the dual problem. In this paper we propose and analyze, as a remedy, both a standard finite element method with mesh grading and a dual variant of the singular complement method. The error order $1/2$ is retained in both cases also with non-convex domains. Numerical experiments confirm the theoretical results.

Key Words Elliptic boundary value problem, very weak formulation, finite element method, mesh grading, singular complement method, discretization error estimate

AMS subject classification 65N30; 65N15

1 Introduction

In this paper we consider the boundary value problem

$$-\Delta y = f \quad \text{in } \Omega, \quad y = u \quad \text{on } \Gamma = \partial\Omega, \quad (1.1)$$

with right hand side $f \in H^{-1}(\Omega)$ and boundary data $u \in L^2(\Gamma)$. We assume $\Omega \subset \mathbb{R}^2$ to be a bounded polygonal domain with boundary Γ . Such problems arise in optimal

*This paper is an extension of our previous paper [1]. The work was partially supported by Deutsche Forschungsgemeinschaft, IGDK 1754.

[†]thomas.apel@unibw.de, Institut für Mathematik und Bauinformatik, Universität der Bundeswehr München, D-85579 Neubiberg, Germany

[‡]snicaise@univ-valenciennes.fr, LAMAV, Institut des Sciences et Techniques de Valenciennes, Université de Valenciennes et du Hainaut Cambrésis, B.P. 311, 59313 Valenciennes Cedex, France

[§]pfefferer@ma.tum.de, Lehrstuhl für Optimalsteuerung, Technische Universität München, Boltzmannstr. 3, D-85748 Garching bei München, Germany

control when the Dirichlet boundary control is considered in $L^2(\Gamma)$ only, see for example [18, 20, 24].

For boundary data $u \in L^2(\Gamma)$ we cannot expect a weak solution $y \in H^1(\Omega)$. Therefore we define a very weak solution by the method of transposition which goes back at least to Lions and Magenes [23]: Find

$$y \in L^2(\Omega) : \quad (y, \Delta v)_\Omega = (u, \partial_n v)_\Gamma - (f, v)_\Omega \quad \forall v \in V \quad (1.2)$$

with $(w, v)_G := \int_G wv$ denoting the $L^2(G)$ scalar product or an appropriate duality product. In our previous paper [2] we showed that the appropriate space V for the test functions is

$$V := H_\Delta^1(\Omega) \cap H_0^1(\Omega) \quad \text{with} \quad H_\Delta^1(\Omega) := \{v \in H^1(\Omega) : \Delta v \in L^2(\Omega)\}. \quad (1.3)$$

In particular it ensures $\partial_n v \in L^2(\Gamma)$ for $v \in V$ such that the formulation (1.2) is well defined. We proved the existence of a unique solution $y \in L^2(\Omega)$ for $u \in L^2(\Gamma)$ and $f \in H^{-1}(\Omega)$, and that the solution is even in $H^{1/2}(\Omega)$. The method of transposition is used in different variants also in [20, 5, 11, 10, 18, 24].

Consider now the discretization of the boundary value problem. Let \mathcal{T}_h be a family of quasi-uniform, conforming finite element meshes, and introduce the finite element spaces

$$Y_h := \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1 \quad \forall T \in \mathcal{T}_h\}, \quad Y_{0h} := Y_h \cap H_0^1(\Omega), \quad Y_h^\partial := Y_h|_{\partial\Omega}.$$

Since the boundary datum u is in general not contained in Y_h^∂ we have to approximate it by $u^h \in Y_h^\partial$, e.g. by using $L^2(\Gamma)$ -projection or quasi-interpolation. In this way, the boundary datum is even regularized since $u^h \in H^{1/2}(\Gamma)$. Hence we can consider a regularized (weak) solution in $Y_*^h := \{v \in H^1(\Omega) : v|_\Gamma = u^h\}$,

$$y^h \in Y_*^h : \quad (\nabla y^h, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega). \quad (1.4)$$

The finite element solution y_h is now searched in $Y_{*h} := Y_*^h \cap Y_h$ and is defined in the classical way: find

$$y_h \in Y_{*h} : \quad (\nabla y_h, \nabla v_h)_\Omega = (f, v_h)_\Omega \quad \forall v_h \in Y_{0h}. \quad (1.5)$$

The same discretization was derived previously by Berggren [5] from a different point of view. In [2] we showed that the discretization error estimate

$$\|y - y_h\|_{L^2(\Omega)} \leq Ch^s \left(h^{1/2} \|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right)$$

holds for $s = 1/2$ if the domain is convex; this is a slight improvement of the result of Berggren, and the convex case is completely treated. In the case of non-convex domains this convergence order is reduced although the very weak solution y is also in $H^{1/2}(\Omega)$; the finite element method does not lead to the best approximation in $L^2(\Omega)$. In order to describe the result we assume for simplicity that Ω has only one corner with interior angle $\omega \in (\pi, 2\pi)$. We proved in [2] the convergence order $s = \lambda - 1/2 - \varepsilon$, where

$\lambda := \pi/\omega$ and $\varepsilon > 0$ arbitrarily small, and showed by numerical experiments that the order of almost $\lambda - 1/2$ is sharp. Note that $s \rightarrow 0$ for $\omega \rightarrow 2\pi$. This is the state of the art for this kind of problem, and our aim is to devise methods to retain the convergence order $s = 1/2$ in the non-convex case.

In order to explain the reduction in the convergence order and our first remedy, let us first mention that we have to modify the Aubin-Nitsche method to derive $L^2(\Omega)$ -error estimates. The first reason is that our problem has no weak solution, only the dual problem,

$$v_z \in V : \quad (\varphi, \Delta v_z)_\Omega = (z, \varphi)_\Omega \quad \forall \varphi \in L^2(\Omega) \quad (1.6)$$

has. The second reason is that the solution y has inhomogeneous Dirichlet data such that an estimate of the $L^2(\Gamma)$ -interpolation error of $\partial_n v_z$ is needed. The $H^1(\Omega)$ -error of a standard finite element method is of order one in convex domains but reduces to $s = \lambda - \varepsilon$ in the case of non-convex domains; moreover, the order of the $L^2(\Gamma)$ -interpolation error of $\partial_n v_z$ reduces from $1/2$ to $\lambda - 1/2 - \varepsilon$. It is known for a long time that locally refined (graded) meshes and augmenting of the finite element space by singular functions are appropriate to retain the optimal convergence order for such problems, see, e. g., [4, 7, 12, 25, 27, 28]. We use these strategies in this paper.

The novelty is that the adapted methods act now implicitly and occur essentially in the analysis for the dual problem. This sounds particularly simple in the case of mesh grading. However, the convergence proof in [2] contains not only interpolation error estimates for the dual solution and its normal derivative (which are improved now) but also the application of an inverse inequality which gives a too pessimistic result if used unchanged in the case of graded meshes. We prove in Section 2 a sharp result by using a weighted norm in intermediate steps. Note we suggest a strong mesh grading with grading parameter $\mu \rightarrow 0$ (the parameter is explained in Section 2) for $\omega \rightarrow 2\pi$ because of the interpolation error estimate of $\partial_n v_z$; the numerical tests show that weaker grading is not sufficient.

The basic idea of the dual singular function method, see [7], or the singular complement method, see [12], is to augment the approximation space for the solution by one (or more, if necessary) singular function of type $r^\lambda \sin(\lambda\theta)$ and the space of test functions by a dual function of type $r^{-\lambda} \sin(\lambda\theta)$, where r, θ are polar coordinates at the concave corner. In this paper we do it the other way round and compute an approximate solution

$$z_h \in Y_h \oplus \text{Span}\{r^{-\lambda} \sin(\lambda\theta)\},$$

such that the error estimate

$$\|y - z_h\|_{L^2(\Omega)} \leq Ch^{1/2} \left(h^{1/2} \|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right)$$

can be shown. Note that the original singular complement method augments the standard finite element space with a function which is part of the representation of the solution. Here, we complement the finite element space with $r^{-\lambda} \sin(\lambda\theta) \notin H^{1/2}(\Omega)$, and although $y \in H^{1/2}(\Omega)$ this has an effect on the approximation order in the $L^2(\Omega)$ -norm.

This makes the method different from the original singular complement method, [12], and we call it *dual singular complement method*. Numerical experiments in Section 4 confirm the theoretical results.

Finally in this introduction, we would like to note that higher order finite elements are not useful here since the solution has low regularity. The extension of our methods to three-dimensional domains should be possible in the case of mesh grading (at considerable technical expenses in the analysis) but is not straightforward in the case of the dual singular complement method since the space $V \setminus H^2(\Omega)$ is in general not finite dimensional, see [13, 14] for the Fourier singular complement method to treat special domains. Curved boundaries could be treated at the prize of using non-affine finite elements, see, e. g., [6, 8, 18].

2 Graded meshes

Recall from the introduction that $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain with boundary Γ , and we consider here the case that Ω has exactly one corner (called *singular corner*) with interior angle $\omega \in (\pi, 2\pi)$. The convex case was already treated in [2] and the case of more than one non-convex corners can be treated similarly since corner singularities are local phenomena.

Without loss of generality we can assume that the singular corner is located at the origin of the coordinate system, and that one boundary edge is contained in the positive x_1 -axis. We recall from [21, 22] that the weak solution of the boundary value problem

$$-\Delta v = g \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \Gamma = \partial\Omega, \quad (2.1)$$

with $g \in L^2(\Omega)$ is not contained in $H^2(\Omega)$ but in

$$H_{\Delta}^1(\Omega) \cap H_0^1(\Omega) = (H^2(\Omega) \cap H_0^1(\Omega)) \oplus \text{Span}\{\xi(r) r^{\lambda} \sin(\lambda\theta)\}, \quad (2.2)$$

ξ being a cut-off function, while r and θ denote polar coordinates at the singular corner.

Let the finite element mesh $\mathcal{T}_h = \{T\}$ be graded with the mesh grading parameter $\mu \in (0, 1]$, i. e., the element size $h_T = \text{diam } T$ and the distance r_T of the element T to the singular corner are related by

$$\begin{aligned} c_1 h^{1/\mu} &\leq h_T \leq c_2 h^{1/\mu} && \text{for } r_T = 0, \\ c_1 h r_T^{1-\mu} &\leq h_T \leq c_2 h r_T^{1-\mu} && \text{for } r_T > 0. \end{aligned} \quad (2.3)$$

Define the finite element spaces

$$Y_h = \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}, \quad Y_{0h} = Y_h \cap H_0^1(\Omega), \quad Y_h^{\partial} = Y_h|_{\partial\Omega}, \quad (2.4)$$

and let the regularized boundary datum $u^h \in Y_h^{\partial} \subset H^{1/2}(\Gamma)$ be defined by the $L^2(\Gamma)$ -projection $\Pi_h u$ or by the Carstensen interpolant $C_h u$, see [9]. To define the latter let \mathcal{N}_{Γ} be the set of nodes of the triangulation on the boundary, and set

$$C_h u = \sum_{x \in \mathcal{N}_{\Gamma}} \pi_x(u) \lambda_x \quad \text{with} \quad \pi_x(u) = \frac{\int_{\Gamma} u \lambda_x}{\int_{\Gamma} \lambda_x} = \frac{(u, \lambda_x)_{\Gamma}}{(1, \lambda_x)_{\Gamma}},$$

where λ_x is the standard hat function related to x . As already outlined in [2], the advantages of the interpolant in comparison with the L^2 -projection are its local definition and the property

$$u \in [a, b] \quad \Rightarrow \quad C_h u \in [a, b],$$

see [17]; a disadvantage may be that $C_h u_h \neq u_h$ for piecewise linear u_h . With these regularized boundary data we define the regularized weak solution $y^h \in Y_*^h := \{v \in H^1(\Omega) : v|_\Gamma = u^h\}$ by (1.4).

Lemma 2.1. *The effect of the regularization of the boundary datum can be estimated by*

$$\|y - y^h\|_{L^2(\Omega)} \leq ch^{1/2} \left(\|u\|_{L^2(\Omega)} + h^{1/2} \|f\|_{H^{-1}(\Omega)} \right)$$

if the mesh is graded with parameter $\mu < 2\lambda - 1$.

Proof. In view of

$$\|y - y^h\|_{L^2(\Omega)} = \sup_{z \in L^2(\Omega), z \neq 0} \frac{(y - y^h, z)_\Omega}{\|z\|_{L^2(\Omega)}}$$

we have to estimate $(y - y^h, z)_\Omega$. To this end, let $z \in L^2(\Omega)$ be an arbitrary function, let $v_z \in V$ be defined by

$$(\varphi, \Delta v_z)_\Omega = (z, \varphi)_\Omega \quad \forall \varphi \in L^2(\Omega), \quad (2.5)$$

see also (1.6). Since the weak regularized solution $y^h \in Y_*^h := \{v \in H^1(\Omega) : v|_\Gamma = u^h\}$ defined by (1.4) is also a very weak solution,

$$(y^h, \Delta v)_\Omega = (u^h, \partial_n v)_\Gamma - (f, v)_\Omega \quad \forall v \in V \quad (2.6)$$

we get with (1.2) and (2.5)

$$(y - y^h, z)_\Omega = (u - u^h, \partial_n v_z)_\Gamma.$$

If u^h is the $L^2(\Gamma)$ -projection $\Pi_h u$ of u we can continue with

$$\begin{aligned} (u - u^h, \partial_n v_z)_\Gamma &= (u - u^h, \partial_n v_z - \Pi_h(\partial_n v_z))_\Gamma = (u, \partial_n v_z - \Pi_h(\partial_n v_z))_\Gamma \\ &\leq \|u\|_{L^2(\Gamma)} \|\partial_n v_z - \Pi_h(\partial_n v_z)\|_{L^2(\Gamma)} \\ &\leq \|u\|_{L^2(\Gamma)} \|\partial_n v_z - C_h(\partial_n v_z)\|_{L^2(\Gamma)} \\ &= \|u\|_{L^2(\Gamma)} \left\| \sum_{x \in \mathcal{N}_\Gamma} (\partial_n v_z - \pi_x(\partial_n v_z)) \lambda_x \right\|_{L^2(\Gamma)} \\ &\leq c \|u\|_{L^2(\Gamma)} \left(\sum_{x \in \mathcal{N}_\Gamma} \|\partial_n v_z - \pi_x(\partial_n v_z)\|_{L^2(\omega_x)}^2 \right)^{1/2}. \end{aligned}$$

If u^h is the Carstensen interpolant of u , there holds

$$\begin{aligned}
(u - C_h u, \partial_n v_z)_\Gamma &= \left(\sum_{x \in \mathcal{N}_\Gamma} (u - \pi_x u) \lambda_x, \partial_n v_z \right)_\Gamma = \sum_{x \in \mathcal{N}_\Gamma} (u - \pi_x(u), (\partial_n v_z) \lambda_x)_\Gamma \\
&= \sum_{x \in \mathcal{N}_\Gamma} (u - \pi_x(u), (\partial_n v_z - \pi_x(\partial_n v_z)) \lambda_x)_\Gamma \\
&\leq \sum_{x \in \mathcal{N}_\Gamma} \|u\|_{L^2(\omega_x)} \|\partial_n v_z - \pi_x(\partial_n v_z)\|_{L^2(\omega_x)} \\
&\leq c \|u\|_{L^2(\Gamma)} \left(\sum_{x \in \mathcal{N}_\Gamma} \|\partial_n v_z - \pi_x(\partial_n v_z)\|_{L^2(\omega_x)}^2 \right)^{1/2},
\end{aligned}$$

i. e., in both cases we have to estimate $\sum_{x \in \mathcal{N}_\Gamma} \|\partial_n v_z - \pi_x(\partial_n v_z)\|_{L^2(\omega_x)}^2$.

To this end we notice that

$$v_z \in V = (H^2(\Omega) \cap H_0^1(\Omega)) \oplus \text{Span}\{\xi(r) r^\lambda \sin(\lambda\theta)\},$$

and consequently

$$\partial_n v_z \in V_\Gamma = \left(\prod_{j=1}^N H_{00}^{1/2}(\Gamma_j) \right) \oplus \text{Span}\{\xi(r) r^{\lambda-1}\},$$

see also the discussion in [2]. This means that we can split $\partial_n v_z = \alpha \xi(r) r^{\lambda-1} + \sum_{j=1}^N w_j$ with $w_j \in H_{00}^{1/2}(\Gamma_j)$ and

$$|\alpha| + \sum_{j=1}^N \|w_j\|_{H_{00}^{1/2}(\Gamma_j)} =: \|\partial_n v_z\|_{V_\Gamma} \leq c \|v_z\|_V := \|\Delta v_z\|_{L^2(\Omega)} = \|z\|_{L^2(\Omega)}.$$

By standard estimates we obtain

$$\left(\sum_{x \in \mathcal{N}_\Gamma} \|w_j - \pi_x w_j\|_{L^2(\omega_x)}^2 \right)^{1/2} \leq ch^{1/2} \|w_j\|_{H_{00}^{1/2}(\Gamma_j)}$$

such that it remains to show that $\left(\sum_{x \in \mathcal{N}_\Gamma} \|\xi(r) r^{\lambda-1} - \pi_x(\xi(r) r^{\lambda-1})\|_{L^2(\omega_x)}^2 \right)^{1/2} \leq ch^{1/2}$

to conclude $\left(\sum_{x \in \mathcal{N}_\Gamma} \|\partial_n v_z - \pi_x(\partial_n v_z)\|_{L^2(\omega_x)}^2 \right)^{1/2} \leq ch^{1/2} \|z\|_{L^2(\Omega)}$.

Denote by $\mathcal{N}_{\Gamma, \text{reg}} \subset \mathcal{N}_\Gamma$ the set of nodes where ω_x does not contain the singular corner. We can estimate

$$\begin{aligned}
\sum_{x \in \mathcal{N}_{\Gamma, \text{reg}}} \|\xi(r) r^{\lambda-1} - \pi_x(\xi(r) r^{\lambda-1})\|_{L^2(\omega_x)}^2 &\leq c \sum_{x \in \mathcal{N}_{\Gamma, \text{reg}}} h_x^2 \|r^{\lambda-2}\|_{L^2(\omega_x)}^2 \\
&\leq ch \sum_{x \in \mathcal{N}_{\Gamma, \text{reg}}} r_x^{1-\mu} r_x \|r^{\lambda-2}\|_{L^2(\omega_x)}^2 \leq ch \int_0^{\text{diam} \Omega} r^{2-\mu+2(\lambda-2)} dr = ch
\end{aligned}$$

for $\mu < 2\lambda - 1$. For the three nodes $x \in \mathcal{N}_\Gamma \setminus \mathcal{N}_{\Gamma, \text{reg}}$ we cannot use the $H^1(\omega_x)$ -regularity of $r^{\lambda-1}$ but there holds simply

$$\|\xi(r) r^{\lambda-1} - \pi_x(\xi(r) r^{\lambda-1})\|_{L^2(\omega_x)} \leq c \|r^{\lambda-1}\|_{L^2(\omega_x)} \sim h_x^{\lambda-1} h_x^{1/2} \sim h^{(\lambda-1/2)/\mu} \sim h^{1/2}$$

for $\mu < 2\lambda - 1$. This finishes the proof. \square

We consider now a lifting $\tilde{B}_h u^h \in Y_{*h}$ defined by the nodal values as follows:

$$(\tilde{B}_h u^h)(x) = \begin{cases} u^h(x), & \text{for all nodes } x \in \Gamma, \\ 0 & \text{for all nodes } x \in \Omega. \end{cases} \quad (2.7)$$

The function y^h and its finite element approximation $y_h \in Y_{*h} = Y_*^h \cap Y_h$ are now defined by

$$y^h = y_f + \tilde{B}_h u^h + \tilde{y}_0^h \quad \text{as well as} \quad y_h = y_{fh} + \tilde{B}_h u^h + \tilde{y}_{0h}, \quad (2.8)$$

where $y_f, \tilde{y}_0^h \in H_0^1(\Omega)$ and $y_{fh}, \tilde{y}_{0h} \in Y_{0h}$ satisfy

$$(\nabla y_f, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega), \quad (2.9)$$

$$(\nabla y_{fh}, \nabla v_h)_\Omega = (f, v_h)_\Omega \quad \forall v_h \in Y_{0h}. \quad (2.10)$$

$$(\nabla \tilde{y}_0^h, \nabla v)_\Omega = -(\nabla(\tilde{B}_h u^h), \nabla v)_\Omega \quad \forall v \in H_0^1(\Omega), \quad (2.11)$$

$$(\nabla \tilde{y}_{0h}, \nabla v_h)_\Omega = -(\nabla(\tilde{B}_h u^h), \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h}. \quad (2.12)$$

In order to estimate $\|y^h - y_h\|_{L^2(\Omega)}$ we estimate $\|y_f - y_{fh}\|_{L^2(\Omega)}$ and $\|\tilde{y}_0^h - \tilde{y}_{0h}\|_{L^2(\Omega)}$.

Lemma 2.2. *The error in approximating y_f satisfies*

$$\|y_f - y_{fh}\|_{L^2(\Omega)} \leq ch \|f\|_{H^{-1}(\Omega)}.$$

if the mesh is graded with parameter $\mu < \lambda$.

Note that the condition $\mu < \lambda$ is weaker than the condition $\mu < 2\lambda - 1$ from Lemma 2.1 since $\lambda < 1$.

Proof. As in the proof of Lemma 2.1, let $z \in L^2(\Omega)$ be an arbitrary function, let $v_z \in V$ be defined via (2.5), and let $v_{zh} \in Y_{0h}$ be the Ritz projection of v_z . By the definitions (2.9) and (2.10) and using the Galerkin orthogonality we get

$$\begin{aligned} (y_f - y_{fh}, z)_\Omega &= (\nabla(y_f - y_{fh}), \nabla v_z)_\Omega = (\nabla(y_f - y_{fh}), \nabla(v_z - v_{zh}))_\Omega \\ &= (\nabla y_f, \nabla(v_z - v_{zh}))_\Omega \leq \|\nabla y_f\|_{L^2(\Omega)} \|\nabla(v_z - v_{zh})\|_{L^2(\Omega)} \end{aligned}$$

By using standard a priori estimates we obtain with grading $\mu < \lambda$

$$\begin{aligned} \|\nabla y_f\|_{L^2(\Omega)} &\leq \|f\|_{H^{-1}(\Omega)}, \\ \|\nabla(v_z - v_{zh})\|_{L^2(\Omega)} &\leq ch \|z\|_{L^2(\Omega)}, \end{aligned}$$

and hence with

$$\|y_f - y_{fh}\|_{L^2(\Omega)} = \sup_{z \in L^2(\Omega), z \neq 0} \frac{(y_f - y_{fh}, z)_\Omega}{\|z\|_{L^2(\Omega)}}$$

the assertion of the lemma. \square

In order to estimate $\|\tilde{y}_0^h - \tilde{y}_{0h}\|_{L^2(\Omega)}$, we divide the domain Ω into subsets Ω_J , i.e.,

$$\Omega = \bigcup_{J=0}^I \Omega_J,$$

where $\Omega_J := \{x : d_{J+1} \leq |x| \leq d_J\}$ for $J = 1, \dots, I-1$, $\Omega_I := \{x : |x| \leq d_I\}$ and $\Omega_0 := \Omega \setminus \bigcup_{J=1}^I \Omega_J$. The radii d_J are set to 2^{-J} and the index I is chosen such that

$$d_I = 2^{-I} = c_I h^{1/\mu} \quad (2.13)$$

with a constant $c_I > 1$ exactly specified later on. In addition we define the extended domains Ω'_J and Ω''_J by

$$\Omega'_J := \Omega_{J-1} \cup \Omega_J \cup \Omega_{J+1} \quad \text{and} \quad \Omega''_J := \Omega'_{J-1} \cup \Omega'_J \cup \Omega'_{J+1},$$

respectively, with the obvious modifications for $J = 0, 1$ and $J = I-1, I$.

Lemma 2.3. *With $\sigma := r + d_I$ there holds the estimate*

$$\|\sigma^{(1-\mu)/2} \nabla \tilde{y}_0^h\|_{L^2(\Omega)} \leq c h^{-1/2} \|u\|_{L^2(\Gamma)}.$$

Proof. We start by rearranging terms, i.e.,

$$\begin{aligned} \|\sigma^{(1-\mu)/2} \nabla \tilde{y}_0^h\|_{L^2(\Omega)}^2 &= \int_{\Omega} \sigma^{1-\mu} \nabla \tilde{y}_0^h \cdot \nabla \tilde{y}_0^h \\ &= \int_{\Omega} \nabla \tilde{y}_0^h \cdot \nabla (\tilde{y}_0^h \sigma^{1-\mu}) - \int_{\Omega} \tilde{y}_0^h \nabla \tilde{y}_0^h \cdot \nabla \sigma^{1-\mu}. \end{aligned} \quad (2.14)$$

For the first term in (2.14) we conclude according to (2.11)

$$\begin{aligned} \int_{\Omega} \nabla \tilde{y}_0^h \cdot \nabla (\tilde{y}_0^h \sigma^{1-\mu}) &= - \int_{\Omega} \nabla (\tilde{B}_h u^h) \cdot \nabla (\tilde{y}_0^h \sigma^{1-\mu}) \\ &= - \int_{\Omega} \sigma^{1-\mu} \nabla (\tilde{B}_h u^h) \cdot \nabla \tilde{y}_0^h - \int_{\Omega} \tilde{y}_0^h \nabla (\tilde{B}_h u^h) \cdot \nabla \sigma^{1-\mu} \\ &\leq \|\sigma^{(1-\mu)/2} \nabla (\tilde{B}_h u^h)\|_{L^2(\Omega)} \left(\|\sigma^{(1-\mu)/2} \nabla \tilde{y}_0^h\|_{L^2(\Omega)} + \|\sigma^{(-1-\mu)/2} \tilde{y}_0^h\|_{L^2(\Omega)} \right), \end{aligned} \quad (2.15)$$

where we used the Cauchy-Schwarz inequality and

$$\nabla \sigma^{1-\mu} = (1-\mu) \sigma^{-\mu} (\cos \theta, \sin \theta)^T. \quad (2.16)$$

Having in mind the decomposition of the domain in subdomains Ω_J , an application of the Poincaré inequality yields for the latter term in (2.15)

$$\begin{aligned} \|\sigma^{(-1-\mu)/2} \tilde{y}_0^h\|_{L^2(\Omega)}^2 &= \sum_{J=0}^I \|\sigma^{(-1-\mu)/2} \tilde{y}_0^h\|_{L^2(\Omega_J)} \|\sigma^{(-1-\mu)/2} \tilde{y}_0^h\|_{L^2(\Omega_J)} \\ &\leq \sum_{J=0}^I d_J^{(-1-\mu)/2} \|\tilde{y}_0^h\|_{L^2(\Omega_J)} \|\sigma^{(-1-\mu)/2} \tilde{y}_0^h\|_{L^2(\Omega_J)} \\ &\leq c \sum_{J=0}^I d_J^{(1-\mu)/2} \|\nabla \tilde{y}_0^h\|_{L^2(\Omega'_J)} \|\sigma^{(-1-\mu)/2} \tilde{y}_0^h\|_{L^2(\Omega_J)} \\ &\leq c \|\sigma^{(1-\mu)/2} \nabla \tilde{y}_0^h\|_{L^2(\Omega)} \|\sigma^{(-1-\mu)/2} \tilde{y}_0^h\|_{L^2(\Omega)}, \end{aligned}$$

where we used $d_J \sim \sigma$ for $x \in \Omega'_J$ twice and the discrete Cauchy-Schwarz inequality. Consequently, we get from (2.15)

$$\int_{\Omega} \nabla \tilde{y}_0^h \cdot \nabla (\tilde{y}_0^h \sigma^{1-\mu}) \leq c \|\sigma^{(1-\mu)/2} \nabla (\tilde{B}_h u^h)\|_{L^2(\Omega)} \|\sigma^{(1-\mu)/2} \nabla \tilde{y}_0^h\|_{L^2(\Omega)}. \quad (2.17)$$

Similarly to the above steps, we get for the second term in (2.14) by means of (2.16)

$$\begin{aligned} \int_{\Omega} \tilde{y}_0^h \nabla \tilde{y}_0^h \cdot \nabla \sigma^{1-\mu} &\leq \|\sigma^{(1-\mu)/2} \nabla \tilde{y}_0^h\|_{L^2(\Omega)} \|\sigma^{(-1-\mu)/2} \tilde{y}_0^h\|_{L^2(\Omega)} \\ &\leq \|\sigma^{(1-\mu)/2} \nabla \tilde{y}_0^h\|_{L^2(\Omega)} \left(\|\sigma^{(-1-\mu)/2} (\tilde{y}_0^h + \tilde{B}_h u^h)\|_{L^2(\Omega)} + \|\sigma^{(-1-\mu)/2} \tilde{B}_h u^h\|_{L^2(\Omega)} \right), \end{aligned} \quad (2.18)$$

such that we infer from (2.14), (2.17) and (2.18) that

$$\begin{aligned} \|\sigma^{(1-\mu)/2} \nabla \tilde{y}_0^h\|_{L^2(\Omega)} &\leq c \left(\|\sigma^{(-1-\mu)/2} \tilde{B}_h u^h\|_{L^2(\Omega)} + \|\sigma^{(1-\mu)/2} \nabla (\tilde{B}_h u^h)\|_{L^2(\Omega)} \right. \\ &\quad \left. + \|\sigma^{(-1-\mu)/2} (\tilde{y}_0^h + \tilde{B}_h u^h)\|_{L^2(\Omega)} \right). \end{aligned} \quad (2.19)$$

Due to the definition of \tilde{B}_h and the definition of the element size h_T in case of graded meshes we easily obtain by means of the norm equivalence in finite dimensional spaces that

$$\|\sigma^{(-1-\mu)/2} \tilde{B}_h u^h\|_{L^2(\Omega)} + \|\sigma^{(1-\mu)/2} \nabla (\tilde{B}_h u^h)\|_{L^2(\Omega)} \leq ch^{-1/2} \|u^h\|_{L^2(\Gamma)} \leq ch^{-1/2} \|u\|_{L^2(\Gamma)}, \quad (2.20)$$

where we employed the stability of u^h in $L^2(\Gamma)$ in the last step. Having in mind the definition (2.13) of d_I , we conclude by applying [2, Lemma 2.8] together with [2, Remark 2.7] that

$$\begin{aligned} \|\sigma^{(-1-\mu)/2} (\tilde{y}_0^h + \tilde{B}_h u^h)\|_{L^2(\Omega)} &\leq d_I^{-\mu/2} \|\sigma^{-1/2} (\tilde{y}_0^h + \tilde{B}_h u^h)\|_{L^2(\Omega)} \\ &\leq ch^{-1/2} \|r^{-1/2} (\tilde{y}_0^h + \tilde{B}_h u^h)\|_{L^2(\Omega)} \leq ch^{-1/2} \|u^h\|_{L^2(\Gamma)} \leq ch^{-1/2} \|u\|_{L^2(\Gamma)}, \end{aligned} \quad (2.21)$$

where we used again the stability of u^h . The estimates (2.19), (2.20) and (2.21) end the proof. \square

Lemma 2.4. *Let $\sigma := r + d_I$ and $\mu \in (0, 2\lambda - 1)$. Then there is the estimate*

$$\|\sigma^{-(1-\mu)/2} (\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega)} \leq ch^{1/2} \|u\|_{L^2(\Gamma)}.$$

Proof. Let $v \in H_0^1(\Omega)$ be the weak solution of

$$-\Delta v = \sigma^{-(1-\mu)} (\tilde{y}_0^h - \tilde{y}_{0h}) \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma,$$

which, according to Theorem 2.15 of [16], has the regularity $v \in V_{(1-\mu)/2}^{2,2}(\Omega)$ (as $\mu < 2\lambda - 1$) and hence $\frac{1}{2}(1-\mu) > 1-\lambda$) and satisfies the a priori estimate

$$|v|_{V_{(1-\mu)/2}^{2,2}(\Omega)} \leq c \|\sigma^{-(1-\mu)} (\tilde{y}_0^h - \tilde{y}_{0h})\|_{V_{(1-\mu)/2}^{0,2}(\Omega)} \leq c \|\sigma^{-(1-\mu)/2} (\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega)}, \quad (2.22)$$

where we use the weighted Sobolev space $V_\beta^{k,2}(\Omega) := \{v \in \mathcal{D}' : \|v\|_{V_\beta^{k,2}(\Omega)} < \infty\}$ with

$$\|v\|_{V_\beta^{k,2}(\Omega)}^2 := \sum_{j=1}^k |v|_{V_{\beta-k+j}^{j,2}(\Omega)}^2, \quad |v|_{V_\beta^{j,2}(\Omega)} := \|r^\beta \nabla^j v\|_{L^2(\Omega)}.$$

Then we obtain by using integration by parts and the Galerkin orthogonality

$$\begin{aligned} \|\sigma^{-(1-\mu)/2}(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega)}^2 &= (\tilde{y}_0^h - \tilde{y}_{0h}, -\Delta v)_\Omega \\ &= (\nabla(\tilde{y}_0^h - \tilde{y}_{0h}), \nabla(v - I_h v))_\Omega = \sum_{J=0}^I (\nabla(\tilde{y}_0^h - \tilde{y}_{0h}), \nabla(v - I_h v))_{\Omega_J} \\ &\leq \sum_{J=0}^I \|\nabla(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega_J)} \|\nabla(v - I_h v)\|_{L^2(\Omega_J)}. \end{aligned} \quad (2.23)$$

By employing standard interpolation error estimates on graded meshes we obtain for any $\mu \in (0, 1]$

$$\|\nabla(v - I_h v)\|_{L^2(\Omega_J)} \leq c h d_J^{(1-\mu)/2} |v|_{V_{(1-\mu)/2}^{2,2}(\Omega'_J)}, \quad (2.24)$$

where the constant c is independent of c_I , see e.g. [3, Lemma 3.7] or [26, Lemma 3.58]. In fact, the constant is essentially the one appearing in the local, elementwise interpolation error estimate. Note that this kind of independence will be crucial when applying a kick back argument further below.

Local finite element error estimates from [19, Theorem 3.4] yield

$$\begin{aligned} \|\nabla(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega_J)} &\leq c \min_{v_h \in Y_{0h}} \left(\|\nabla(\tilde{y}_0^h - v_h)\|_{L^2(\Omega'_J)} + \frac{1}{d_J} \|\tilde{y}_0^h - v_h\|_{L^2(\Omega'_J)} \right) \\ &\quad + c \frac{1}{d_J} \|\tilde{y}_0^h - \tilde{y}_{0h}\|_{L^2(\Omega'_J)}. \end{aligned}$$

By choosing $v_h \equiv 0$ and by applying the Poincaré inequality, we conclude

$$\begin{aligned} \|\nabla(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega_J)} &\leq c \left(\|\nabla \tilde{y}_0^h\|_{L^2(\Omega'_J)} + \frac{1}{d_J} \|\tilde{y}_0^h - \tilde{y}_{0h}\|_{L^2(\Omega'_J)} \right) \\ &\leq c \left(\|\nabla \tilde{y}_0^h\|_{L^2(\Omega'_J)} + d_J^{(-1-\mu)/2} \|\sigma^{-(1-\mu)/2}(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega'_J)} \right), \end{aligned} \quad (2.25)$$

where we used $d_J \sim \sigma$ for $x \in \Omega'_J$. Consequently, we get from (2.23)–(2.25)

$$\begin{aligned} &\|\sigma^{-(1-\mu)/2}(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega)}^2 \\ &\leq c \sum_{J=0}^I \left(h \|\sigma^{(1-\mu)/2} \nabla \tilde{y}_0^h\|_{L^2(\Omega'_J)} + h d_J^{-\mu} \|\sigma^{-(1-\mu)/2}(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega'_J)} \right) |v|_{V_{(1-\mu)/2}^{2,2}(\Omega'_J)} \\ &\leq c \left(h \|\sigma^{(1-\mu)/2} \nabla \tilde{y}_0^h\|_{L^2(\Omega)} + c_I^{-\mu} \|\sigma^{-(1-\mu)/2}(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega)} \right) |v|_{V_{(1-\mu)/2}^{2,2}(\Omega)}, \end{aligned}$$

where we again employed $d_J \sim \sigma$ for $x \in \Omega_J''$, $hd_J^{-\mu} \leq c_I^{-\mu}$, which holds due to the definition (2.13) of d_I , and the discrete Cauchy-Schwarz inequality. For $\mu \in (0, 2\lambda - 1)$ we infer by the a priori estimate (2.22) that

$$\|\sigma^{-(1-\mu)/2}(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega)} \leq c \left(h \|\sigma^{(1-\mu)/2} \nabla \tilde{y}_0^h\|_{L^2(\Omega)} + c_I^{-\mu} \|\sigma^{-(1-\mu)/2}(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega)} \right).$$

By choosing c_I large enough we can kick back the second term in the above inequality such that Lemma 2.3 yields the desired result. \square

Theorem 2.5. *For $\mu \in (0, 2\lambda - 1)$ we get*

$$\|y - y_h\|_{L^2(\Omega)} \leq ch^{1/2} \left(\|u\|_{L^2(\Omega)} + h^{1/2} \|f\|_{H^{-1}(\Omega)} \right). \quad (2.26)$$

Proof. Due to the boundedness of $\sigma^{(1-\mu)/2}$ independent of h for all $\mu \in (0, 1]$ we obtain from Lemma 2.4

$$\|\tilde{y}_0^h - \tilde{y}_{0h}\|_{L^2(\Omega)} \leq \|\sigma^{(1-\mu)/2}\|_{L^\infty(\Omega)} \|\sigma^{-(1-\mu)/2}(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega)} \leq ch^{1/2} \|u\|_{L^2(\Gamma)}. \quad (2.27)$$

In view of (2.8) we get by using the triangle inequality

$$\|y - y_h\|_{L^2(\Omega)} \leq \|y - y^h\|_{L^2(\Omega)} + \|y_f - y_{fh}\|_{L^2(\Omega)} + \|\tilde{y}_0^h - \tilde{y}_{0h}\|_{L^2(\Omega)}.$$

These three terms are bounded by the right hand side of (2.26) in Lemmata 2.1 and 2.2 as well as in (2.27). \square

3 The dual singular complement method

3.1 Analytical background and regularization

Using the notation of the previous section, we recall that the splitting (2.2)

$$H_\Delta^1(\Omega) \cap H_0^1(\Omega) = (H^2(\Omega) \cap H_0^1(\Omega)) \oplus \text{Span}\{\xi(r) r^\lambda \sin(\lambda\theta)\},$$

implies that

$$R := \{\Delta v : v \in H^2(\Omega) \cap H_0^1(\Omega)\},$$

is a closed subspace of $L^2(\Omega)$. It is shown in [22, Sect. 2.3] that

$$L^2(\Omega) = R \oplus^\perp \text{Span}\{p_s\}, \quad (3.1)$$

with the *dual singular function*

$$p_s = r^{-\lambda} \sin(\lambda\theta) + \tilde{p}_s \quad (3.2)$$

where $\tilde{p}_s \in H^1(\Omega)$ is chosen such that the decomposition (3.1) is orthogonal for the $L^2(\Omega)$ inner product. Therefore, the dual singular function p_s is a solution of

$$w \in L^2(\Omega) : \quad (\Delta v, w) = 0 \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega), \quad (3.3)$$

which proves the non-uniqueness of the solution of (3.3). This is the dual property to the non-existence of a solution of (2.1) in $H^2(\Omega) \cap H_0^1(\Omega)$, see [22, Introduction].

Due to (3.1) we can split any $L^2(\Omega)$ -function into $L^2(\Omega)$ -orthogonal parts. To this end denote by Π_R and Π_{p_s} the orthogonal projections on R and on $\text{Span}\{p_s\}$, respectively, i.e., for $g \in L^2(\Omega)$, it is $g = \Pi_R g + \Pi_{p_s} g$ where

$$\begin{aligned}\Pi_{p_s} g &= \alpha(g) p_s \quad \text{with} \quad \alpha(g) = \frac{(g, p_s)_\Omega}{\|p_s\|_{L^2(\Omega)}^2}, \\ \Pi_R g &= g - \Pi_{p_s} g.\end{aligned}$$

Since $p_s \in L^2(\Omega)$ there exists

$$\phi_s \in H_\Delta^1(\Omega) \cap H_0^1(\Omega) : \quad -\Delta \phi_s = p_s, \quad (3.4)$$

see also Section 3.3 for more details on ϕ_s . For the moment we assume that p_s and ϕ_s are explicitly known; hence the decomposition $g = \Pi_R g + \alpha(g) p_s$ can be computed once g is given. Computable approximations of p_s and ϕ_s are discussed in Section 3.3.

Now we come back to problem (1.2) and decompose its solution y in the form

$$y = \Pi_R y + \alpha(y) p_s. \quad (3.5)$$

From the decomposition (3.1) we see that problem (1.2) is equivalent to

$$\begin{aligned}(y, p_s)_\Omega &= -(u, \partial_n \phi_s)_\Gamma + (f, \phi_s)_\Omega, \\ (y, \Delta v)_\Omega &= (u, \partial_n v)_\Gamma - (f, v)_\Omega \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega)\end{aligned}$$

and with the orthogonal splitting (3.5) to

$$\begin{aligned}\alpha(y) (p_s, p_s)_\Omega &= -(u, \partial_n \phi_s)_\Gamma + (f, \phi_s)_\Omega, \\ (\Pi_R y, \Delta v)_\Omega &= (u, \partial_n v)_\Gamma - (f, v)_\Omega \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega).\end{aligned}$$

The first equation directly yields $\alpha(y)$, namely

$$\alpha(y) = \frac{-(u, \partial_n \phi_s)_\Gamma + (f, \phi_s)_\Omega}{(p_s, p_s)_\Omega}, \quad (3.6)$$

hence the projection of y on p_s is known. It remains to find an approximation of $\Pi_R y$.

At this point we recall the regularization approach from [2] which we summarized already in the introduction. Let $u^h \in H^{1/2}(\Gamma)$ be a regularized boundary datum (this can be any, e.g. $\Pi_h u$ or $C_h u$ from Section 2, but we do not assume graded meshes here) such that we can define the regularized (weak) solution in $Y_*^h := \{v \in H^1(\Omega) : v|_\Gamma = u^h\}$,

$$y^h \in Y_*^h : \quad (\nabla y^h, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega). \quad (3.7)$$

In [2, Remark 2.13] we showed that the regularization error can be estimated by

$$\|y - y^h\|_{L^2(\Omega)} \leq c \|u - u^h\|_{H^{-s}(\Gamma)}$$

where $0 < s < \lambda - \frac{1}{2}$ (if Ω was convex we would get $s = \frac{1}{2}$, that means the regularization error is in general bigger in the non-convex case). With the next lemma we show that $\Pi_R(y - y^h)$ is not affected by non-convex corners.

Lemma 3.1. *The estimate*

$$\|\Pi_R(y - y^h)\|_{L^2(\Omega)} \leq C\|u - u^h\|_{H^{-1/2}(\Gamma)}$$

holds.

Proof. Recall $V = H_\Delta^1(\Omega) \cap H_0^1(\Omega)$ from (1.3). From (3.7) and the Green formula, we have for any $v \in V$

$$(f, v)_\Omega = (\nabla y^h, \nabla v)_\Omega = -(y^h, \Delta v)_\Omega + (y^h, \partial_n v)_\Gamma.$$

Note that $v \in V$ is sufficient, see [15, Lemma 3.4]. Subtracting this expression from the very weak formulation (1.2), we get

$$(y - y^h, \Delta v)_\Omega = (u - u^h, \partial_n v)_\Gamma \quad \forall v \in V.$$

Restricting this identity to $v \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$(\Pi_R(y - y^h), \Delta v)_\Omega = (u - u^h, \partial_n v)_\Gamma \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \quad (3.8)$$

Now for any $z \in R$, we let $v_z \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique solution of

$$\Delta v_z = z, \quad (3.9)$$

that satisfies

$$\|\partial_n v_z\|_{H^{1/2}(\Gamma)} \leq c\|v_z\|_{H^2(\Omega)} \leq c\|z\|_{L^2(\Omega)}. \quad (3.10)$$

Since for any $g \in L^2(\Omega)$ the equality

$$(\Pi_R(y - y^h), g)_\Omega = (\Pi_R(y - y^h), \Pi_R g)_\Omega = (y - y^h, \Pi_R g)_\Omega$$

holds we get with (3.8)–(3.10)

$$\begin{aligned} \|\Pi_R(y - y^h)\|_{L^2(\Omega)} &= \sup_{z \in R, z \neq 0} \frac{(y - y^h, z)_\Omega}{\|z\|_{L^2(\Omega)}} = \sup_{z \in R, z \neq 0} \frac{(u - u^h, \partial_n v_z)_\Gamma}{\|z\|_{L^2(\Omega)}} \\ &\leq \|u - u^h\|_{H^{-1/2}(\Gamma)} \sup_{z \in R, z \neq 0} \frac{\|\partial_n v_z\|_{H^{1/2}(\Gamma)}}{\|z\|_{L^2(\Omega)}} \leq c\|u - u^h\|_{H^{-1/2}(\Gamma)} \end{aligned}$$

which is the estimate to be proved. \square

3.2 Discretization by standard finite elements

Recall from (2.4) the finite element spaces

$$Y_h = \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}, \quad Y_{0h} = Y_h \cap H_0^1(\Omega), \quad Y_h^\partial = Y_h|_{\partial\Omega},$$

defined now on a family \mathcal{T}_h of quasi-uniform, conforming finite element meshes. Assume that the regularized boundary datum u^h is contained in Y_h^∂ such that the estimates

$$\|u^h\|_{L^2(\Gamma)} \leq c\|u\|_{L^2(\Gamma)}, \quad (3.11)$$

$$\|u - u^h\|_{H^{-1/2}(\Gamma)} \leq Ch^{1/2}\|u\|_{L^2(\Gamma)}, \quad (3.12)$$

hold. It can be derived from [2, Lemma 2.14] that this can be accomplished by using the $L^2(\Gamma)$ -projection or by quasi-interpolation. A consequence of Lemma 3.1 is the estimate

$$\|\Pi_R(y - y^h)\|_{L^2(\Omega)} \leq Ch^{1/2}\|u\|_{L^2(\Gamma)}. \quad (3.13)$$

(In the case of a convex domain the operator Π_R is the identity, and the corresponding error estimates were already proven in [2].)

As already done in the introduction, define further the finite element solution $y_h \in Y_{*h} := Y_*^h \cap Y_h$ via

$$y_h \in Y_{*h} : \quad (\nabla y_h, \nabla v_h)_\Omega = (f, v_h)_\Omega \quad \forall v_h \in Y_{0h}. \quad (3.14)$$

We proved in [2] that in the case of quasi-uniform meshes \mathcal{T}_h

$$\|y - y_h\|_{L^2(\Omega)} \leq Ch^s \left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right) \quad (3.15)$$

holds for $s \in (0, \lambda - \frac{1}{2})$ (again $s = \frac{1}{2}$ for convex domains). As before, in the next lemma we show that $\Pi_R(y - y_h)$ is not affected by the non-convex corners.

Lemma 3.2. *The discretization error estimate*

$$\|\Pi_R(y - y_h)\|_{L^2(\Omega)} \leq Ch^{1/2} \left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right)$$

holds.

Proof. By the triangle inequality we have

$$\|\Pi_R(y - y_h)\|_{L^2(\Omega)} \leq \|\Pi_R(y - y^h)\|_{L^2(\Omega)} + \|\Pi_R(y^h - y_h)\|_{L^2(\Omega)}. \quad (3.16)$$

The first term is estimated in (3.13). For the second term we first notice that $y^h - y_h \in H_0^1(\Omega)$ satisfies the Galerkin orthogonality

$$(\nabla(y^h - y_h), \nabla v_h)_\Omega = 0 \quad \forall v_h \in Y_{0h}, \quad (3.17)$$

see (1.4) and (1.5). With that, we estimate $\|\Pi_R(y^h - y_h)\|_{L^2(\Omega)}$ by a similar arguments as $\|\Pi_R(y - y^h)\|_{L^2(\Omega)}$ in the proof of Lemma 3.1. Recall from (3.9) and (3.10) that $v_z \in H^2(\Omega) \cap H_0^1(\Omega)$ is the weak solution of $\Delta v_z = z \in R$. It can be approximated by the Lagrange interpolant $I_h v_z$ satisfying

$$\|\nabla(v_z - I_h v_z)\|_{L^2(\Omega)} \leq ch\|v_z\|_{H^2(\Omega)} \leq ch\|z\|_{L^2(\Omega)}.$$

We get

$$\begin{aligned}
\|\Pi_R(y^h - y_h)\|_{L^2(\Omega)} &= \sup_{z \in R, z \neq 0} \frac{(y^h - y_h, z)_\Omega}{\|z\|_{L^2(\Omega)}} = \sup_{z \in R, z \neq 0} \frac{(\nabla(y^h - y_h), \nabla v_z)_\Omega}{\|z\|_{L^2(\Omega)}} \\
&= \sup_{z \in R, z \neq 0} \frac{(\nabla(y^h - y_h), \nabla(v_z - I_h v_z))_\Omega}{\|z\|_{L^2(\Omega)}} \\
&\leq ch \|\nabla(y^h - y_h)\|_{L^2(\Omega)}. \tag{3.18}
\end{aligned}$$

In order to bound $\|\nabla(y^h - y_h)\|_{L^2(\Omega)}$ by the data we consider the lifting $\tilde{B}_h u^h \in Y_{*h}$ defined by (2.7). The next steps are simpler than in Section 2 since we have quasi-uniform meshes and obtain a sharp estimate also by using an inverse inequality below. The homogenized solution $y_0^h = y^h - \tilde{B}_h u^h \in H_0^1(\Omega)$ satisfies

$$(\nabla y_0^h, \nabla v)_\Omega = (f, v)_\Omega - (\nabla(\tilde{B}_h u^h), \nabla v)_\Omega \quad \forall v \in H_0^1(\Omega).$$

By taking $v = y_0^h$ we see that

$$\|\nabla y_0^h\|_{L^2(\Omega)}^2 \leq \|f\|_{H^{-1}(\Omega)} \|y_0^h\|_{H^1(\Omega)} + \|\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)} \|\nabla y_0^h\|_{L^2(\Omega)}.$$

Using the Poincaré inequality we obtain

$$\|\nabla y_0^h\|_{L^2(\Omega)} \leq c \|f\|_{H^{-1}(\Omega)} + \|\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)}, \tag{3.19}$$

and with the Céa lemma

$$\begin{aligned}
\|\nabla(y^h - y_h)\|_{L^2(\Omega)} &\leq \|\nabla(y^h - \tilde{B}_h u^h)\|_{L^2(\Omega)} = \|\nabla y_0^h\|_{L^2(\Omega)} \\
&\leq c \|f\|_{H^{-1}(\Omega)} + \|\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)}.
\end{aligned}$$

The remaining term $\|\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)}$ is estimated by using the inverse inequality

$$\|\nabla(\tilde{B}_h u^h)\|_{L^2(T)} \leq ch^{-1/2} \|u^h\|_{L^2(E)}.$$

for $E \subset T \cap \Gamma$, $T \in \mathcal{T}_h$, which can be proved by standard scaling arguments, to get

$$\|\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)} \leq ch^{-1/2} \|u^h\|_{L^2(\Gamma)}. \tag{3.20}$$

Hence we proved

$$\|\nabla(y^h - y_h)\|_{L^2(\Omega)} \leq c \|f\|_{H^{-1}(\Omega)} + ch^{-1/2} \|u^h\|_{L^2(\Gamma)}.$$

With (3.16), (3.13), (3.18), the previous inequality, and (3.11) we finish the proof. \square

With (3.5) we can immediately conclude the following result.

Corollary 3.3. *Let $y_h \in Y_{*h}$ be the solution of (3.14), then the discretization error estimate*

$$\|y - (\Pi_R y_h + \alpha(y) p_s)\|_{L^2(\Omega)} \leq Ch^{1/2} \left(h^{1/2} \|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right)$$

holds, reminding that p_s and $\alpha(y)$ are given by (3.2) and (3.6), respectively.

Hence the positive result is that $\Pi_R y_h + \alpha(y) p_s$ is a better approximation of y than y_h . The problem is that p_s and ϕ_s are used explicitly, and in practice they are not known. A remedy of this drawback is the aim of the next section.

3.3 Approximate singular functions

Following [12], we approximate p_s from (3.2) by

$$\begin{aligned} p_s^h &= p_h^* - r_h + r^{-\lambda} \sin(\lambda\theta), \quad r_h = \tilde{B}_h \left(r^{-\lambda} \sin(\lambda\theta) \right), \\ p_h^* &\in Y_{0h} : \quad (\nabla p_h^*, \nabla v_h)_\Omega = (\nabla r_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h}, \end{aligned} \quad (3.21)$$

with \tilde{B}_h from (2.7). The function ϕ_s from (3.4) admits the splitting

$$\phi_s = \tilde{\phi} + \beta r^\lambda \sin(\lambda\theta), \quad (3.22)$$

with $\tilde{\phi} \in H^2(\Omega)$ and $\beta = \pi^{-1} \|p_s\|_{L^2(\Omega)}^2$, see again [12]. It is approximated by

$$\begin{aligned} \phi_s^h &= \phi_h^* - \beta_h s_h + \beta_h r^\lambda \sin(\lambda\theta), \quad s_h = \tilde{B}_h \left(r^\lambda \sin(\lambda\theta) \right), \quad \beta_h = \frac{1}{\pi} \|p_s^h\|_{L^2(\Omega)}^2, \\ \phi_h^* &\in Y_{0h} : \quad (\nabla \phi_h^*, \nabla v_h)_\Omega = (p_s^h, v_h)_\Omega + \beta_h (\nabla s_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h}, \end{aligned} \quad (3.23)$$

that means, $\tilde{\phi}$ is approximated by $\tilde{\phi}_h = \phi_h^* - \beta_h s_h \in Y_h$. The approximation errors are bounded by

$$\|p_s - p_s^h\|_{L^2(\Omega)} \leq ch^{2\lambda-\epsilon} \leq ch, \quad (3.24)$$

$$|\beta - \beta_h| \leq ch^{2\lambda-\epsilon} \leq ch, \quad (3.25)$$

$$\|\phi_s - \phi_s^h\|_{1,\Omega} \leq ch, \quad (3.26)$$

see [12, Lemmas 3.1–3.3], where (3.25) and (3.26) imply

$$\|\tilde{\phi} - \tilde{\phi}_h\|_{1,\Omega} \leq ch. \quad (3.27)$$

At the end of Section 3.2 we saw that $\Pi_R y_h + \alpha(y)p_s$ is a better approximation of y than y_h . Since this function is not computable we approximate it by

$$z_h = \Pi_R^h y_h + \alpha_h p_s^h, \quad (3.28)$$

with

$$\Pi_R^h y_h = y_h - \gamma_h p_s^h, \quad \gamma_h = \frac{(y_h, p_s^h)_\Omega}{\|p_s^h\|_{L^2(\Omega)}^2} \quad (3.29)$$

and a suitable approximation α_h of

$$\alpha(y) = \frac{-(u, \partial_n \phi_s)_\Gamma + (f, \phi_s)_\Omega}{(p_s, p_s)_\Omega}$$

from (3.6). To this end we write the problematic term by using (3.22) as

$$(u, \partial_n \phi_s)_\Gamma = (u, \partial_n \tilde{\phi})_\Gamma + \beta (u, \partial_n (r^\lambda \sin(\lambda\theta)))_\Gamma.$$

and replace the term $(u, \partial_n \tilde{\phi})_\Gamma$ by $(u^h, \partial_n \tilde{\phi})_\Gamma$. Since $\tilde{\phi}$ belongs to $H^2(\Omega)$ and u^h is the trace of $\tilde{B}_h u^h$, we get by using the Green formula

$$\begin{aligned} (u^h, \partial_n \tilde{\phi})_\Gamma &= (\tilde{B}_h u^h, \Delta \tilde{\phi})_\Omega + (\nabla \tilde{B}_h u^h, \nabla \tilde{\phi})_\Omega \\ &= -(\tilde{B}_h u^h, p_s)_\Omega + (\nabla \tilde{B}_h u^h, \nabla \tilde{\phi})_\Omega \end{aligned} \quad (3.30)$$

as $\Delta \tilde{\phi} = \Delta \phi_s = -p_s$. With all these notations and results, we define

$$\alpha_h = \frac{(\tilde{B}_h u^h, p_s^h)_\Omega - (\nabla \tilde{B}_h u^h, \nabla \tilde{\phi}_h)_\Omega - \beta_h(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma + (f, \phi_s^h)_\Omega}{(p_s^h, p_s^h)_\Omega^2}. \quad (3.31)$$

Note that α_h can be computed explicitly and therefore z_h as well.

Let us estimate the approximation errors made.

Lemma 3.4. *Let $y_h \in Y_{*h}$ be the solution of (3.14). Then the error estimates*

$$\|\Pi_R y_h - \Pi_R^h y_h\|_{L^2(\Omega)} \leq ch \left(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right), \quad (3.32)$$

$$|\alpha(y) - \alpha_h| \leq ch^{1/2} \left(h^{1/2} \|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right) \quad (3.33)$$

hold.

Proof. With the definitions of Π_R and Π_R^h , with $\gamma := (y_h, p_s)_\Omega / \|p_s\|_{L^2(\Omega)}^2$, and by using the triangle inequality we have

$$\begin{aligned} \|\Pi_R y_h - \Pi_R^h y_h\|_{L^2(\Omega)} &= \|\gamma p_s - \gamma_h p_s^h\|_{L^2(\Omega)} \\ &\leq |\gamma - \gamma_h| \|p_s^h\|_{L^2(\Omega)} + |\gamma| \|p_s - p_s^h\|_{L^2(\Omega)} \end{aligned}$$

We write

$$\begin{aligned} \gamma - \gamma_h &= \frac{(y_h, p_s)_\Omega}{\|p_s\|_{L^2(\Omega)}^2} - \frac{(y_h, p_s^h)_\Omega}{\|p_s^h\|_{L^2(\Omega)}^2} \\ &= \frac{(y_h, p_s - p_s^h)_\Omega}{\|p_s\|_{L^2(\Omega)}^2} + (y_h, p_s^h)_\Omega \left(\frac{1}{\|p_s\|_{L^2(\Omega)}^2} - \frac{1}{\|p_s^h\|_{L^2(\Omega)}^2} \right) \\ &= \frac{(y_h, p_s - p_s^h)_\Omega}{\|p_s\|_{L^2(\Omega)}^2} + (y_h, p_s^h)_\Omega \frac{(p_s^h + p_s, p_s^h - p_s)_\Omega}{\|p_s\|_{L^2(\Omega)}^2 \|p_s^h\|_{L^2(\Omega)}^2}, \end{aligned}$$

and by the Cauchy-Schwarz inequality and (3.24) we get

$$|\gamma - \gamma_h| \leq ch \|y_h\|_{L^2(\Omega)}.$$

We have used that $\|p_s\|_{L^2(\Omega)}$ and $\|p_s^h\|_{L^2(\Omega)}$ can be treated as constants due to the definition of p_s and due to (3.24). We conclude with $|\gamma| \leq c \|y_h\|_{L^2(\Omega)}$, and (3.24) that

$$\|\Pi_R y_h - \Pi_R^h y_h\|_{L^2(\Omega)} \leq ch \|y_h\|_{L^2(\Omega)}. \quad (3.34)$$

In view of the finite element error estimate (3.15) and the standard a priori estimate for the very weak solution,

$$\|y\|_{L^2(\Omega)} \leq c \left(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right),$$

see Lemma 2.3 of [2], we have

$$\|y_h\|_{L^2(\Omega)} \leq \|y\|_{L^2(\Omega)} + \|y - y_h\|_{L^2(\Omega)} \leq c \left(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right).$$

This estimate together with (3.34) proves (3.32).

The proof of the estimate (3.33) is based on writing the problematic term in the definition of $\alpha(y)$ without approximation as

$$\begin{aligned} (u, \partial_n \phi_s)_\Gamma &= (u, \partial_n \tilde{\phi})_\Gamma + \beta(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma \\ &= (u - u^h, \partial_n \tilde{\phi})_\Gamma + (u^h, \partial_n \tilde{\phi})_\Gamma + \beta(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma \\ &= (u - u^h, \partial_n \tilde{\phi})_\Gamma - (\tilde{B}_h u^h, p_s)_\Omega + (\nabla \tilde{B}_h u^h, \nabla \tilde{\phi})_\Omega + \beta(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma \end{aligned}$$

where we used (3.30) in the last step. Consequently, we showed that

$$\begin{aligned} \alpha(y) - \alpha_h &= \frac{1}{\|p_s\|_{L^2(\Omega)}^2} \left(- (u - u^h, \partial_n \tilde{\phi})_\Gamma + (\tilde{B}_h u^h, p_s - p_s^h)_\Omega - (\nabla \tilde{B}_h u^h, \nabla(\tilde{\phi} - \tilde{\phi}^h))_\Omega \right. \\ &\quad \left. - (\beta - \beta_h)(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma + (f, \phi_s - \phi_s^h)_\Omega \right). \end{aligned}$$

To prove (3.33), in view of (3.24), (3.25), and (3.26) it remains to show that

$$\begin{aligned} \left| (u - u^h, \partial_n \tilde{\phi})_\Gamma \right| &\leq ch^{1/2} \|u\|_{L^2(\Gamma)}, \\ \left| (\tilde{B}_h u^h, p_s - p_s^h)_\Omega \right| &\leq ch^{1/2} \|u\|_{L^2(\Gamma)}, \\ \left| (\nabla \tilde{B}_h u^h, \nabla(\tilde{\phi} - \tilde{\phi}^h))_\Omega \right| &\leq ch^{1/2} \|u\|_{L^2(\Gamma)}. \end{aligned}$$

The first estimate follows from the estimate (3.12) and the fact that $\tilde{\phi}$ belongs to $H^2(\Omega)$. The second one follows from the Cauchy-Schwarz inequality and the estimates (3.20) and (3.24). Similarly, the third estimate follows from the Cauchy-Schwarz inequality and the estimates (3.20) and (3.27). \square

Corollary 3.5. *Let Ω be a non-convex domain and let $y_h \in Y_{*h}$ be the solution of (3.14) and let z_h be derived by (3.28), (3.29), and (3.31), then the discretization error estimate*

$$\|y - z_h\|_{L^2(\Omega)} \leq Ch^{1/2} \left(h^{1/2} \|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)} \right)$$

holds.

Proof. The main ingredients of the proof were already derived. Indeed, it is

$$\begin{aligned}\|y - z_h\|_{L^2(\Omega)} &= \|\Pi_R y + \alpha(y)p_s - \Pi_R^h y_h - \alpha_h p_s^h\|_{L^2(\Omega)} \\ &\leq \|\Pi_R y - \Pi_R y_h\|_{L^2(\Omega)} + \|\Pi_R y_h - \Pi_R^h y_h\|_{L^2(\Omega)} + \\ &\quad |\alpha(y) - \alpha_h| \|p_s\|_{L^2(\Omega)} + |\alpha_h| \|p_s - p_s^h\|_{L^2(\Omega)}.\end{aligned}$$

The first three terms can be estimated by using Lemmas 3.2 and 3.4. So it remains to treat the fourth term. To bound $|\alpha_h|$ we use the triangle inequality

$$|\alpha_h| \leq |\alpha_h - \alpha(y)| + |\alpha(y)|.$$

For the first term we use (3.33), while for the second term we use (3.6) reminding that ϕ_s belongs to $H^{3/2+\epsilon}(\Omega)$ with some $\epsilon > 0$. Altogether we have

$$|\alpha_h| \leq C (\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)})$$

and conclude by using (3.24). \square

3.4 The method in form of an algorithm

Before we describe the numerical experiments, let us summarize the algorithm.

1. Compute the finite element solution

$$y_h \in Y_{*h} : \quad (\nabla y_h, \nabla v_h)_\Omega = (f, v_h)_\Omega \quad \forall v_h \in Y_{0h}$$

where $Y_{*h} = \{v_h \in Y_h : v_h|_\Gamma = u^h\}$, compare (1.5), with $u^h \in Y_h^\partial$ being an approximation of the boundary datum u satisfying (3.11) and (3.12).

2. Compute the approximate singular functions:

$$r_h = \tilde{B}_h \left(r^{-\lambda} \sin(\lambda\theta) \right),$$

$$p_h^* \in Y_{0h} : \quad (\nabla p_h^*, \nabla v_h)_\Omega = (\nabla r_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h},$$

$$\tilde{p}_h = p_h^* - r_h,$$

$$\beta_h = \frac{1}{\pi} \|\tilde{p}_h + r^{-\lambda} \sin(\lambda\theta)\|_{L^2(\Omega)}^2,$$

$$s_h = \tilde{B}_h \left(r^\lambda \sin(\lambda\theta) \right),$$

$$\phi_h^* \in Y_{0h} : \quad (\nabla \phi_h^*, \nabla v_h)_\Omega = (\tilde{p}_h + r^{-\lambda} \sin(\lambda\theta), v_h)_\Omega + \beta_h (\nabla s_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h},$$

$$\tilde{\phi}_h = \phi_h^* - \beta_h s_h,$$

compare (3.21) and (3.23).

3. Compute

$$\begin{aligned}\gamma_h &= \frac{(y_h, p_s^h)_\Omega}{(p_s^h, p_s^h)_\Omega} \quad \text{with } p_s^h = \tilde{p}_h + r^{-\lambda} \sin(\lambda\theta), \\ \alpha_h &= \frac{(\tilde{B}_h u^h, p_s^h)_\Omega - (\nabla \tilde{B}_h u^h, \nabla \tilde{\phi}_h)_\Omega - \beta_h(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma + (f, \phi_s^h)_\Omega}{(p_s^h, p_s^h)_\Omega}, \\ \delta_h &= \alpha_h - \gamma_h, \\ \tilde{z}_h &= y_h + \delta_h \tilde{p}_h,\end{aligned}$$

compare (3.29) and (3.31). According to (3.28), the numerical solution is

$$z_h = \tilde{z}_h + \delta_h r^{-\lambda} \sin(\lambda\theta).$$

Note that all integrals with r^λ and $r^{-\lambda}$ must be computed with care.

4 Numerical experiment

This section is devoted to the numerical verification of our theoretical results. For that purpose we present an example with known solution. Furthermore, to examine the influence of the corner singularities, we consider several polygonal domain Ω_ω depending on an interior angle $\omega \in (0, 2\pi)$; we present here the results for $\omega = 270^\circ$ and $\omega = 355^\circ$. The computational domains are defined by

$$\Omega_\omega := (-1, 1)^2 \cap \{x \in \mathbb{R}^2 : (r(x), \theta(x)) \in (0, \sqrt{2}] \times [0, \omega]\}, \quad (4.1)$$

where r and θ stand for the polar coordinates located at the origin. The boundary of Ω_ω is denoted by Γ_ω . We solve the problem

$$-\Delta y = 0 \quad \text{in } \Omega_\omega, \quad y = u \quad \text{on } \Gamma_\omega, \quad (4.2)$$

numerically by using a standard finite element method with graded meshes and the proposed dual singular function method with quasi-uniform meshes. The boundary datum u is chosen to be

$$u := r^{-0.4999} \sin(-0.4999\theta) \quad \text{on } \Gamma_\omega.$$

This function belongs to $L^p(\Gamma)$ for every $p < 2.0004$. The exact solution of our problem is simply

$$y = r^{-0.4999} \sin(-0.4999\theta),$$

since y is harmonic.

Quasi-uniform finite element meshes are generated from a coarse initial mesh by using a newest vertex bisection algorithm. Graded meshes are generated by marking and bisecting elements until the grading condition (2.3) is fulfilled with suitable constants c_1 and c_2 , see Figure 1. As a regularization we have used the $L^2(\Gamma)$ -projection. The discretization errors are calculated by an adaptive quadrature formula.

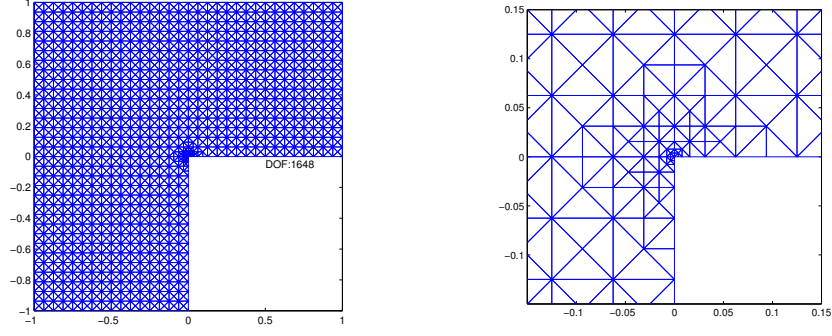


Figure 1: Graded mesh with $\mu = 0.3333$, generated by newest vertex bisection; left: whole mesh, right: zoom

unknowns	standard	eoc	DSCM	eoc
33	0.736		0.653	
113	0.645	0.215	0.587	0.154
417	0.568	0.193	0.423	0.472
1601	0.503	0.181	0.303	0.482
6273	0.447	0.175	0.216	0.489
24833	0.397	0.171	0.154	0.493
98817	0.353	0.169	0.109	0.496
394241	0.314	0.168	0.077	0.498
expected		0.167		0.5

Table 1: Discretization errors $e_h = y - y_h$ with quasi-uniform mesh (standard) and $e_h = y - z_h$ (DSCM) for $\omega = 270^\circ$

The discretization errors for different mesh sizes and the corresponding experimental orders of convergence are given in Tables 1 and 2 for the interior angle $\omega = 270^\circ$ and in Tables 3 and 4 for the interior angle $\omega = 355^\circ$. We see that the numerical results confirm the expected convergence rate $1/2$ for the dual singular complement method and the finite element method on sufficiently graded meshes. For $\mu > 2\lambda - 1$ we obtain a convergence rate of about $(\lambda - 1/2)/\mu$ only which can certainly be proven with an adaption of the techniques used in Section 2 but is of less interest. We show the numerical results here mainly to underline that the strong grading $\mu < 2\lambda - 1$ is indeed necessary for optimal convergence.

Finally, in Figures 2 and 3 we display the exact and some computed solutions for a visual comparison. There is a pole of type $r^{-0.4999}$ in the boundary data and hence in the exact solution. The standard finite element solution and the solution on graded meshes are computed after regularization of the boundary datum which replaces the infinite value for $r = 0$ by a finite one, which may be big as in the case of $\omega = 355^\circ$. One can also see that the behavior for $r \rightarrow 0$ can be approximated better with graded meshes. The solution with the DSCM contains two parts, a the finite element function on a

$\mu = 0.666$			$\mu = 0.5$			$\mu = 0.333$		
unknowns	error	eoc	unknowns	error	eoc	unknowns	error	eoc
33	0.736		33	0.736		33	0.736	
113	0.645	0.215	113	0.645	0.215	113	0.645	0.215
421	0.498	0.392	424	0.505	0.369	428	0.445	0.559
1618	0.446	0.165	1631	0.398	0.354	1648	0.312	0.524
6343	0.348	0.361	6381	0.315	0.344	6463	0.220	0.512
25111	0.314	0.153	25244	0.249	0.339	25544	0.155	0.508
99881	0.246	0.354	100423	0.198	0.336	101563	0.110	0.504
398436	0.221	0.150	400553	0.157	0.335	405014	0.077	0.502
expected		0.25			0.333			0.5

Table 2: Discretization errors $e_h = y - y_h$ for $\omega = 270^\circ$

unknowns	standard	eoc	DSCM	eoc
46	1.105		1.010	
159	1.069	0.053	1.021	
589	1.049	0.029	0.834	0.291
2265	1.036	0.018	0.590	0.500
8881	1.028	0.012	0.417	0.500
35169	1.021	0.010	0.295	0.499
139969	1.015	0.008	0.209	0.497
558465	1.010	0.008	0.148	0.495
expected		0.007		0.5

Table 3: Discretization errors $e_h = y - y_h$ with quasi-uniform mesh (standard) and $e_h = y - z_h$ (DSCM) for $\omega = 355^\circ$

$\mu = 0.5$			$\mu = 0.3$			$\mu = 0.014085$		
unknowns	error	eoc	unknowns	error	eoc	unknowns	error	eoc
46	1.105		46	1.105		46	1.105	
159	1.069	0.053	159	1.069	0.053	159	1.069	0.047
597	1.039	0.044	602	1.031	0.055	970	0.854	0.325
2301	1.023	0.023	2335	1.012	0.028	4116	0.600	0.509
9014	1.011	0.017	9166	0.990	0.032	16154	0.424	0.502
35682	1.001	0.015	36197	0.975	0.022	62949	0.298	0.508
141986	0.991	0.015	144015	0.962	0.020	247276	0.210	0.505
566419	0.981	0.014	574296	0.942	0.030	979316	0.148	0.505
expected		0.014			0.023			0.5

Table 4: Discretization errors $e_h = y - z_h$ for $\omega = 355^\circ$

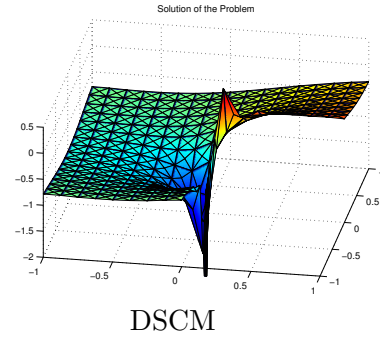
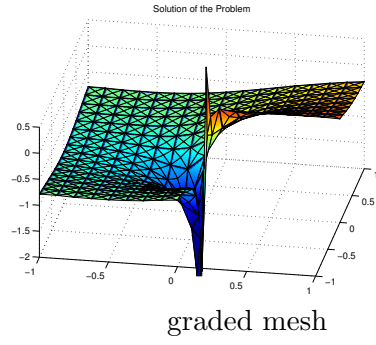
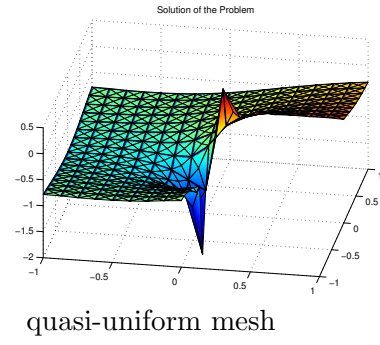
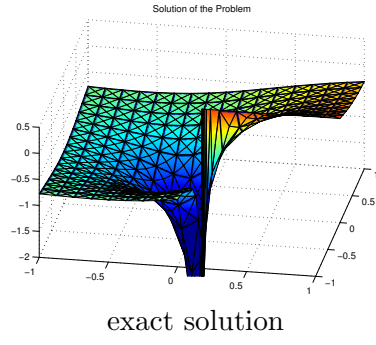


Figure 2: Visual comparison of the solutions, $\omega = 270^\circ$

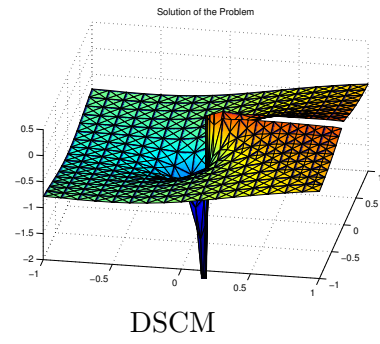
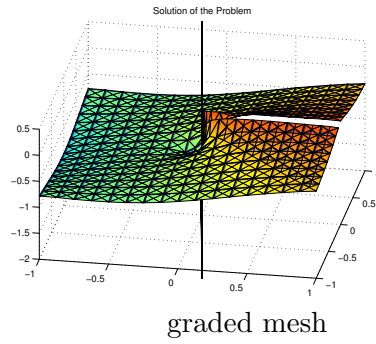
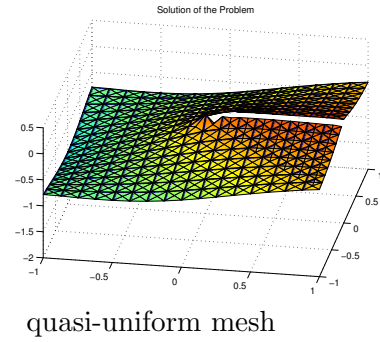
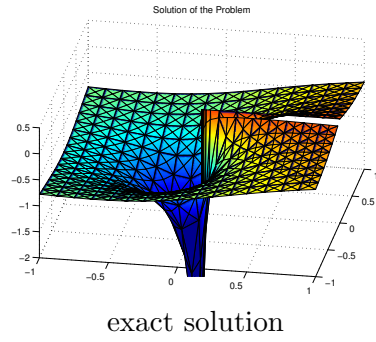


Figure 3: Visual comparison of the solutions, $\omega = 355^\circ$

quasi-uniform mesh and a multiple of the singular function $r^{-\lambda} \sin(\lambda\theta)$ which has a pole of type $r^{-2/3}$ for $\omega = 270^\circ$ and of type $r^{-180/355}$ for $\omega = 355^\circ$. The latter term produces an infinite value for $r = 0$ and has a asymptotic behaviour which is different from the exact solution. Interesting enough, the $L^2(\Omega)$ -error of the DSCM solution profits from the presence of this term.

Concerning the DSCM, we emphasize that the quadrature formula for the numerical evaluation of the integral

$$(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma$$

has to be adapted in order to get a sufficiently good approximation. Otherwise, the error due to quadrature dominates the overall error. In our implementation, we chose for the numerical integration a graded mesh on the boundary ($h_E \sim hr_E^{1-\mu}$ if the distance r_E of the boundary edge E satisfies $0 < r_E < R$ with R being the radius of the refinement zone and μ being the refinement parameter, and $h_T = h^{1/\mu}$ for $r_E = 0$) combined with a one-point Gauss quadrature rule on each element. The choice $\mu \leq 2\pi/\omega - 1$ seems to be the correct grading to achieve a convergence order of $1/2$. For the results presented in Tables 1 and 3 we used $R = 0.1$ and $\mu = 2\pi/\omega - 1$.

References

- [1] T. Apel, S. Nicaise, and J. Pfefferer. A dual singular complement method for the numerical solution of the Poisson equation with L^2 boundary data in non-convex domains. Preprint arXiv:1505.00414 [math.NA], arXiv, 2015.
- [2] T. Apel, S. Nicaise, and J. Pfefferer. Discretization of the Poisson equation with non-smooth data and emphasis on non-convex domains. To appear in Numer. Methods Partial Differential Equations, 2016.
- [3] T. Apel, J. Pfefferer, and A. Rösch. Finite element error estimates on the boundary with application to optimal control. *Math. Comp.*, 84:33–70, 2015.
- [4] I. Babuška. Error-bounds for finite element method. *Numerische Mathematik*, 16:322–333, 1970/1971.
- [5] M. Berggren. Approximations of very weak solutions to boundary-value problems. *SIAM J. Numer. Anal.*, 42(2):860–877, 2004.
- [6] C. Bernardi. Optimal finite-element interpolation on curved domains. *SIAM J. Numer. Anal.*, 26(5):1212–1240, 1989.
- [7] H. Blum and M. Dobrowolski. On finite element methods for elliptic equations on domains with corners. *Computing*, 28(1):53–63, 1982.
- [8] J. H. Bramble and J. T. King. A robust finite element method for nonhomogeneous Dirichlet problems in domains with curved boundaries. *Math. Comp.*, 63(207):1–17, 1994.

- [9] C. Carstensen. Quasi-interpolation and a posteriori error analysis in finite element methods. *M2AN, Math. Model. Numer. Anal.*, 33(6):1187–1202, 1999.
- [10] E. Casas, M. Mateos, and J.-P. Raymond. Penalization of Dirichlet optimal control problems. *ESAIM. Control, Optimisation and Calculus of Variations*, 15(4):782–809, 2009.
- [11] E. Casas and J.-P. Raymond. Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations. *SIAM J. Control Optim.*, 45(5):1586–1611, 2006.
- [12] P. Ciarlet, Jr. and J. He. The singular complement method for 2d scalar problems. *C. R. Math. Acad. Sci. Paris*, 336(4):353–358, 2003.
- [13] P. Ciarlet, Jr., B. Jung, S. Kaddouri, S. Labrunie, and J. Zou. The Fourier singular complement method for the Poisson problem. I. Prismatic domains. *Numer. Math.*, 101(3):423–450, 2005.
- [14] P. Ciarlet, Jr., B. Jung, S. Kaddouri, S. Labrunie, and J. Zou. The Fourier singular complement method for the Poisson problem. II. Axisymmetric domains. *Numer. Math.*, 102(4):583–610, 2006.
- [15] M. Costabel. Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.*, 19(3):613–626, 1988.
- [16] M. Dauge, S. Nicaise, M. Bourlard, and J. M.-S. Lubuma. Coefficients des singularités pour des problèmes aux limites elliptiques sur un domaine à points coniques. I. Résultats généraux pour le problème de Dirichlet. *RAIRO Modél. Math. Anal. Numér.*, 24(1):27–52, 1990.
- [17] J. C. de los Reyes, C. Meyer, and B. Vexler. Finite element error analysis for state-constrained optimal control of the Stokes equations. *Control Cybernet.*, 37(2):251–284, 2008.
- [18] K. Deckelnick, A. Günther, and M. Hinze. Finite element approximation of Dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains. *SIAM J. Control Optim.*, 48(4):2798–2819, 2009.
- [19] A. Demlow, J. Guzmán, and A. H. Schatz. Local energy estimates for the finite element method on sharply varying grids. *Math. Comp.*, 80(273):1–9, 2011.
- [20] D. A. French and J. T. King. Approximation of an elliptic control problem by the finite element method. *Numer. Funct. Anal. Optimization*, 12(3-4):299–314, 1991.
- [21] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman, Boston–London–Melbourne, 1985.
- [22] P. Grisvard. *Singularities in boundary value problems*, volume 22 of *Research Notes in Applied Mathematics*. Springer, New York, 1992.

- [23] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1, 2.* Travaux et Recherches Mathématiques. Dunod, Paris, 1968.
- [24] S. May, R. Rannacher, and B. Vexler. Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems. *SIAM J. Control Optim.*, 51(3):2585–2611, 2013.
- [25] L. A. Oganessian and L. A. Ruhovec. *Variatsionno-raznostnye metody resheniya ellipticheskikh uravnenii.* Akad. Nauk Armyan. SSR, Erevan, 1979.
- [26] J. Pfefferer. *Numerical analysis for elliptic Neumann boundary control problems on polygonal domains.* PhD thesis, Universität der Bundeswehr München, 2014. <http://athene.bibl.unibw-muenchen.de:8081/node?id=92055>.
- [27] G. Raugel. Résolution numérique par une méthode d’éléments finis du problème de Dirichlet pour le laplacien dans un polygone. *C. R. Acad. Sci. Paris, Sér. A*, 286(18):A791–A794, 1978.
- [28] G. Strang and G. J. Fix. *An analysis of the finite element method.* Prentice-Hall, Inc., Englewood Cliffs, N. J., 1973.